MATH 212

Assignment 6

3.4.2 - $FSS[f(x)] = \sum_{k=1}^{\infty} b_k \sin k\pi x$

$FCS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi x$

$a_0 = 2\int_0^1 f(x)\,dx$

$a_k = 2\int_0^1 f(x)\cos k\pi x\,dx$

$b_k = 2\int_0^1 f(x)\sin k\pi x\,dx$

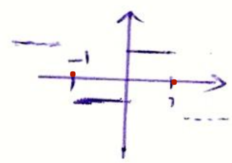
(a) $f(x) = 1$

$a_0 = 2\int_0^1 dx = 2$

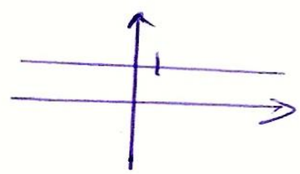
$a_k = 2\int_0^1 \cos k\pi x\,dx = \frac{2}{k\pi}\sin k\pi x\Big]_0^1 = 0$

$b_k = 2\int_0^1 \sin k\pi x\,dx = \frac{-2}{k\pi}\cos k\pi x\Big]_0^1 = \frac{-2}{k\pi}((-1)^k - 1)$

$FSS[f(x)] = \sum_{k=1}^{\infty} \frac{-2}{k\pi}((-1)^k - 1)\sin k\pi x$



$FCS[f(x)] = 1$



(b) $f(x) = \sin\pi x$

$a_0 = 2\int_0^1 \sin\pi x\,dx = \frac{-2}{\pi}\cos\pi x\Big]_0^1 = \frac{-2}{\pi}\left[(-1)^k - 1\right]$

$k \geq 1 \quad a_k = 2\int_0^1 \sin\pi x\cos k\pi x\,dx$ $\quad \sin a\cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

$= \int_0^1 \sin(\pi x + k\pi x) + \sin(\pi x - k\pi x)\,dx$

$= \int_0^1 \sin((1+k)\pi x) + \sin((1-k)\pi x)\,dx$

$* k = 1 \Rightarrow a_1 = \int_0^1 \sin(2\pi x)\,dx = \frac{-1}{2\pi}\cos 2\pi x\Big]_0^1 = 0$

$* k \neq 1 \Rightarrow a_k = \frac{-1}{(1+k)\pi}\cos(1+k)\pi x\Big]_0^1 - \frac{1}{(1-k)\pi}\cos(1-k)\pi x\Big]_0^1$

$$a_k = \frac{-1}{(1+k)\pi}\left[\cos(1+k)\pi - 1\right] - \frac{1}{(1-k)\pi}\left[\cos(1-k)\pi - 1\right]$$

$$= \frac{-1}{(1+k)\pi}\left[(-1)^{k+1} - 1\right] - \frac{1}{(1-k)\pi}\left[(-1)^{k+1} - 1\right]$$

$$= \left[(-1)^{k+2} + 1\right]\left[\frac{1}{(1+k)\pi} - \frac{1}{(1-k)\pi}\right]$$

$$= \frac{-2k\left(1 + (-1)^{k+2}\right)}{\pi(1-k)(1+k)}$$

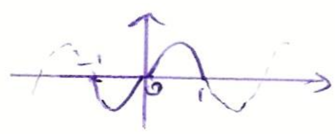
$$b_k = 2\int_0^1 \sin \pi x \sin k\pi x \, dx$$ $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$= \int_0^1 \cos\left[(1-k)\pi x\right] - \cos\left[(1+k)\pi x\right] \, dx$$

$$k = 1 \quad b_k = \int_0^1 1 - \cos 2\pi x \, dx = x - \frac{1}{2\pi}\sin 2\pi x \Big]_0^1 = 1$$

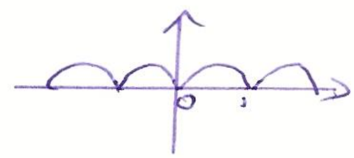
$$k \neq 1 : b_k = \frac{1}{(1-k)\pi}\sin\left((1-k)\pi x\right) - \frac{1}{(1+k)\pi}\sin(1+k)\pi x \Big]_0^1$$

$$= 0$$



$$\therefore \; FSS[f(x)] = \sin \pi x$$

$$FCS[f(x)] = -\frac{1}{\pi}\left[(-1)^k - 1\right] + \sum_{k=2}^{\infty} \frac{-2k\left(1+(-1)^{k+2}\right)}{\pi(1-k)(1+k)} \cos k\pi x$$



(3)

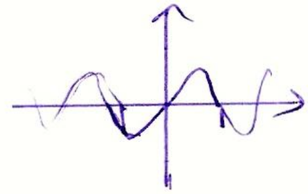
(c) $f(x) = \sin^3 \pi x = \dfrac{\sin \pi x}{2}\left(1 - \cos(2\pi x)\right)$

$$= \frac{\sin \pi x}{2} - \frac{\sin \pi x \cos 2\pi x}{2}$$

$$= \frac{\sin \pi x}{2} - \frac{1}{4}\left[\sin 3\pi x + \sin(-\pi x)\right]$$

$$= \frac{3}{4} \sin \pi x - \frac{1}{4} \sin 3\pi x$$

$$FSS[f(x)] = \frac{3}{4} \sin \pi x - \frac{1}{4} \sin 3\pi x$$



$$a_0 = 2 \int_0^1 \frac{3}{4} \sin \pi x - \frac{1}{4} \sin 3\pi x \, dx$$

$$= \frac{-3}{2\pi}(-1-1) + \frac{1}{6\pi}(-1-1) = \frac{3}{\pi} - \frac{1}{3\pi}$$

$$a_k = 2 \int_0^1 \left(\frac{3}{4} \sin \pi x - \frac{1}{4} \sin 3\pi x\right) \sin k\pi x \, dx$$

$$= \frac{3}{4}\left[\int_0^1 2 \sin \pi x \sin k\pi x \, dx\right] - \frac{1}{2} \int_0^1 \sin 3\pi x \sin k\pi x \, dx$$

check (b)   0 by (b)

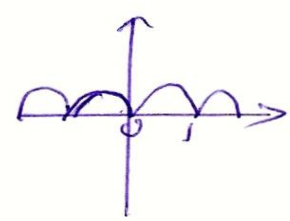
$$k=1, \quad a_1 = \frac{3}{4} - \frac{1}{2} \int_0^1 \sin 3\pi x \sin \pi x \, dx = \frac{3}{4}$$

$$k \neq 1: \quad a_k = 0 - \frac{1}{2} \int_0^1 \sin 3\pi x \sin k\pi x \, dx$$

$$= -\frac{1}{2\pi} \int_0^{\pi} \underbrace{\sin 3X \sin kX}_{\text{even}} \, dX \qquad X = \pi x, \quad dX = \pi \, dx$$

$$= -\frac{1}{\pi} \int_{-\pi}^{\pi} \sin 3X \sin kX \, dX$$

$$= \begin{cases} 0 & k \neq 3 \\ -1 & k = 3 \end{cases}$$



$\boxed{a_3 = -1}$

$$FCS[f(x)] = \frac{3}{2\pi} - \frac{1}{6\pi} + \frac{3}{4} \cos \pi x - \cos 3\pi x$$

(d) $f(x) = x(1-x) = x - x^2$

$$a_0 = 2\int_0^1 (x-x^2)\,dx = \frac{1}{3}$$

$$a_k = 2\int_0^1 (x-x^2)\cos k\pi x\,dx$$

$$= \frac{2(-2)}{(k\pi)^2}(-1)^k = \frac{4}{(k\pi)^2}(-1)^{k+1}$$

$$\begin{array}{lll} x-x^2 & + & \cos k\pi x \\ 1-2x & - & \frac{1}{k\pi}\sin k\pi x \\ -2 & + & \frac{-1}{(k\pi)^2}\cos k\pi x \\ 0 & & \frac{-1}{(k\pi)^3}\sin k\pi x \end{array}$$

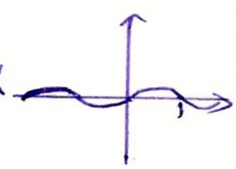
$$b_k = 2\int_0^1 (x-x^2)\sin k\pi x\,dx$$

$$= 2\left[\frac{-2(-1)^k}{(k\pi)^3} + \frac{2}{(k\pi)^3}\right]$$

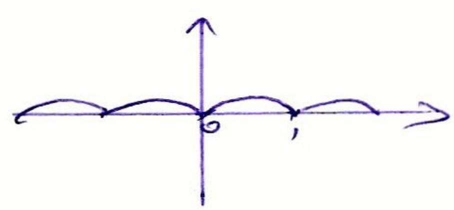
$$= \frac{4}{(k\pi)^3}\left[1+(-1)^{k+1}\right]$$

$$\begin{array}{lll} x-x^2 & + & \sin k\pi x \\ 1-2x & - & \frac{-1}{k\pi}\cos k\pi x \\ -2 & + & \frac{-1}{(k\pi)^2}\sin k\pi x \\ 0 & & \frac{1}{(k\pi)^3}\cos k\pi x \end{array}$$

$$FSS[f(x)] = \sum_{k=1}^{\infty} \frac{4}{(k\pi)^3}\left[1+(-1)^{k+1}\right]\sin k\pi x$$



$$FCS[f(x)] = \frac{1}{6} + \sum_{k=1}^{\infty} \frac{4}{(k\pi)^2}(-1)^{k+1}\cos k\pi x$$



3.4.3. (a) $f(x) = |x| \qquad -3 \le x \le 3 \qquad \ell = 3$

$$a_0 = \frac{1}{3}\int_{-3}^{3} |x|\,dx \underset{\text{even}}{=} \frac{2}{3}\int_0^3 x\,dx = \frac{2}{3}\cdot\frac{x^2}{2}\Big]_0^3 = 3.$$

$$a_k = \frac{1}{3}\int_{-3}^{3} |x| \cos\frac{k\pi x}{3}\,dx \underset{\text{even}}{=} \frac{2}{3}\int_0^3 x\cos\frac{k\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\frac{3x}{k\pi}\sin\frac{k\pi x}{3} + \frac{9}{(k\pi)^2}\cos\frac{k\pi x}{3}\right]_0^3$$

$$\begin{array}{ll} x & \cos\frac{k\pi x}{3} \\ 1 & \frac{3}{k\pi}\sin\frac{k\pi x}{3} \\ 0 & \frac{-9}{(k\pi)^2}\cos\frac{k\pi x}{3} \end{array}$$

$$= \frac{2}{3}\left[\frac{9}{(k\pi)^2}(-1)^k - \frac{9}{(k\pi)^2}\right]$$

$$= \frac{6}{(k\pi)^2}\left[(-1)^k - 1\right]$$

$$b_k = \frac{1}{3}\int_{-3}^{3} \underbrace{|x|\sin\frac{k\pi x}{3}}_{\text{odd}}\,dx = 0.$$

$$FS[|x|] = \frac{3}{2} + \sum_{k=1}^{\infty} \frac{6}{(k\pi)^2}\left[(-1)^k - 1\right]\cos\frac{k\pi x}{3}$$

(b) $f(x) = x^2 - 4 \qquad -2 \le x \le 2 \qquad \ell = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2}(x^2-4)\,dx = \frac{x^3}{3} - 4x\Big]_0^2 = -\frac{16}{3}$$

$$a_k = \frac{1}{2}\int_{-2}^{2}(x^2-4)\cos\frac{k\pi x}{2}\,dx$$

$$= \int_0^2 (x^2-4)\cos\frac{k\pi x}{2}\,dx$$

$$\begin{array}{ll} x^2-4 & \cos\frac{k\pi x}{2} \\ 2x & \frac{2}{k\pi}\sin\frac{k\pi x}{2} \\ 2 & \frac{2^2}{(k\pi)^2}\cos\frac{k\pi x}{2} \\ 0 & \frac{-(2)^3}{(k\pi)^3}\sin\frac{k\pi x}{2} \end{array}$$

$$= \left[\frac{8x}{(k\pi)^2}\cos\frac{k\pi x}{2}\right]_0^2$$

$$= \frac{16}{(k\pi)^2}(-1)^{k+1}$$

$$b_k = \frac{1}{2}\int_{-2}^{2} (x^2-4)\sin\frac{k\pi x}{2}\,dx = 0.$$ 6

$$FS[x^2-4] = -\frac{8}{3} + \sum_{k=1}^{\infty} \frac{16}{(k\pi)^2}(-1)^{k+1}\cos\frac{k\pi x}{2}$$

(c) $f(x) = e^x \quad -10 \le x \le 10 \quad \ell = 10$

$$a_0 = \frac{1}{10}\int_{-10}^{10} e^x\,dx = \frac{1}{10}\left[\frac{e^{10}-e^{-10}}{2}\right] = \frac{1}{10}\sinh 10$$

$$a_k = \frac{1}{10}\int_{-10}^{10} e^x \cos\frac{k\pi x}{10}\,dx = \frac{1}{10}\left[\frac{e^x}{1+\left(\frac{k\pi}{10}\right)^2}\left(\cos\frac{k\pi x}{10} + \frac{k\pi}{10}\sin\frac{k\pi x}{10}\right)\right]_{-10}^{10}$$

$$= \frac{1}{10}\left[\frac{e^{10}(-1)^k}{1+\left(\frac{k\pi}{10}\right)^2} - \frac{e^{-10}}{1+\left(\frac{k\pi}{10}\right)^2}(-1)^k\right]$$

$$= \frac{(-1)^k}{10\left[1+\frac{k^2\pi^2}{10^2}\right]}\cdot 2\sinh 10 = \frac{20(-1)^k}{10^2+k^2\pi^2}\sinh 10.$$

$$b_k = \frac{1}{10}\int_{-10}^{10} e^x \sin\frac{k\pi x}{10}\,dx$$

$$= \frac{1}{10}\left[\frac{e^x}{1+\left(\frac{k\pi}{10}\right)^2}\left(\sin\frac{k\pi x}{10} - \frac{k\pi}{10}\cos\frac{k\pi x}{10}\right)\right]_{-10}^{10}$$

$$= \frac{1}{10}\left[\frac{e^{10}}{1+\left(\frac{k\pi}{10}\right)^2}\cdot\frac{k\pi}{10}\cdot(-1)^{k+1} + \frac{e^{-10}}{1+\left(\frac{k\pi}{10}\right)^2}\cdot\frac{k\pi}{10}(-1)^k\right]$$

$$= \frac{(-1)^{k+1}\cdot k\pi}{10^2+(k\pi)^2}\cdot 2\sinh 10$$

$$= \frac{(-1)^{k+1}\,2k\pi}{100+k^2\pi^2}\sinh 10$$

$$FS[e^x] = \frac{1}{20}\sinh 10 + \sum_{k=1}^{\infty}\frac{20(-1)^k}{10^2+k^2\pi^2}\sinh 10\cos\frac{k\pi x}{10}$$

$$+ \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,2k\pi}{100+k^2\pi^2}\sinh 10\,\sin\frac{k\pi x}{10}.$$

Can use complex Fourier series and then transform it to real

(d) $f(x) = \sin x \quad -1 \leq x \leq 1$

$$a_0 = \int_{-1}^{1} \sin x \, dx = 0$$

$$a_k = \int_{-1}^{1} \sin x \cos k\pi x \, dx = 0$$

$$b_k = \int_{-1}^{1} \sin x \sin k\pi x \, dx$$

$$= \int_0^1 \left[\cos(x - k\pi x) - \cos(x + k\pi x) \right] dx$$

$$= \int_0^1 \cos(1-k\pi)x - \cos(1+k\pi)x \, dx$$

$\sin(a-b) = \sin a \cos b - \cos a \sin a$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$$\overset{\infty}{=} \frac{1}{1-k\pi} \sin(1-k\pi)x - \frac{1}{1+k\pi} \sin(1+k\pi)x \Big]_0^1$$

$$= \frac{1}{1-k\pi} \sin 1 \cos k\pi - \frac{1}{1+k\pi} \sin 1 \cos k\pi$$

$$= (\sin 1)(-1)^k \left[\frac{1}{1-k\pi} - \frac{1}{1+k\pi} \right]$$

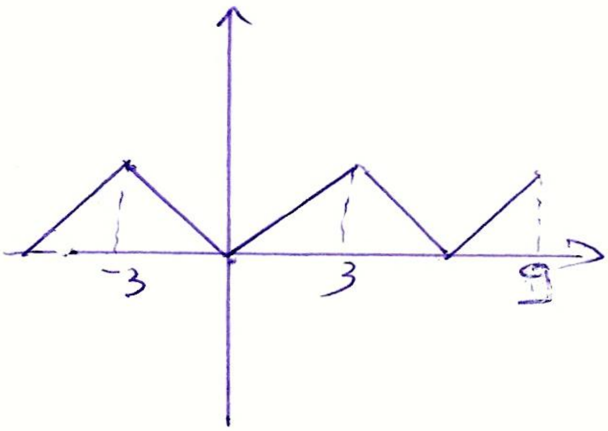
$$= \frac{2k\pi (-1)^k \sin 1}{(1-k\pi)(1+k\pi)}$$

$$\widetilde{FS}[\sin x] = \sum_{k=1}^{\infty} \frac{2k\pi (-1)^k \sin 1}{(1-k\pi)(1+k\pi)} \sin k\pi x$$

Graphs

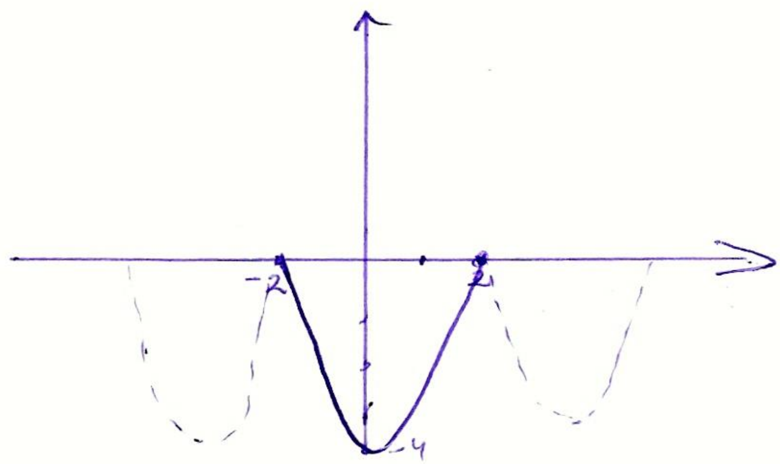
(a) $\tilde{f}(x) = |x - 6m| \qquad 6m-3 \le x \le 6m+3$

$m \in \mathbb{Z}$

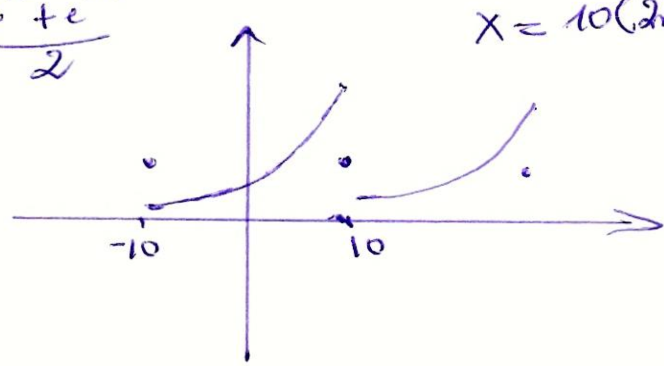


(b) $\tilde{f}(x) = (x-4m)^2 - 4 \qquad 2(2m-1) \le x \le 2(2m+1)$

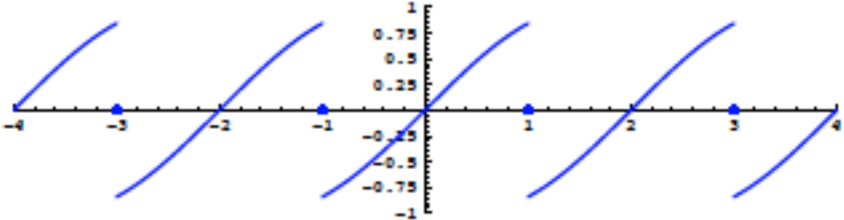
$m \in \mathbb{Z}$



(c) $\tilde{f}(x) = \begin{cases} e^{x-20m} & 10(2m-1) < x < 10(2m+1) \\ \dfrac{e^{10}+e^{-10}}{2} & x = 10(2m+1) \end{cases} \qquad m \in \mathbb{Z}$



(d) $\tilde{f}(x) = \sin(x-2m)$ $\qquad 2m-1 \leq x \leq 2m+1 \quad m \in \mathbb{Z}$


1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
-4
-3
-2
-1
1
2
3
4

3.4.4. (a) $\left[\widetilde{FS}[|x|]\right]' = \sum_{k=1}^{\infty} -\frac{k\pi}{3} \cdot \frac{6}{(k\pi)^2}\left[(-1)^k - 1\right] \sin\frac{k\pi x}{3}$

$= FS\left[\begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}\right]$

Yes, since $\tilde{f}(x)$; the 6-periodic extension of f, is continuous.

(b) $\left[FS[x^2-4]\right]' = \sum_{k=1}^{\infty} -\frac{k\pi}{2} \cdot \frac{16}{(k\pi)^2}(-1)^{k+1} \sin\frac{k\pi x}{2}$

$= FS[2x]$

$$= \sum_{k=1}^{\infty} \frac{-8}{k\pi}(-1)^{k+1} \sin\frac{k\pi x}{2}$$

Yes

$\tilde{f}(x)$ is continuous

(c) $\left[FS[e^x]\right]' = \sum_{k=1}^{\infty} \frac{-20(-1)^k}{10^2+k^2\pi^2} \cdot \frac{k\pi}{10} \sinh 10 \sin\frac{k\pi x}{10}$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, 2k\pi}{10^2+k^2\pi^2} \cdot \frac{k\pi}{10} \sinh 10 \cos\frac{k\pi x}{10}$$

No, since $\tilde{f}(x)$ not continuous

(d) $\left[FS[\sin x]\right]' = \sum_{k=1}^{\infty} \frac{2k^2\pi^2(-1)^k \sin 1}{(1-k\pi)(1+k\pi)} \cos k\pi x$

No, since f~(x) is not continuous

3.4.5.

(10)

(a) $FS\left[\int_0^x |s|\,ds - \frac{3}{2}x\right] = m + \sum_{k=1}^{\infty} \frac{6x^3}{(k\pi)^3}\left[(-1)^k - 1\right] \sin\frac{k\pi x}{3}$

$$m = \frac{1}{2\ell}\int_{-\ell}^{\ell} g(s)\,ds \qquad g(x) = \begin{cases} \frac{1}{2}x^2, & x>0 \\ -\frac{1}{2}x^2, & x<0 \end{cases}$$

$$= \frac{1}{6}\left[\int_{-3}^{0} -\frac{1}{2}s^2\,ds + \int_0^3 \frac{1}{2}s^2\,ds\right]$$

$$= \frac{1}{6}\left[-\frac{1}{6}s^3\right]_{-3}^{0} + \frac{1}{6}\left[\frac{1}{6}s^3\right]_0^3 = 0$$

FS[x] over [-3, 3]

FS[x^2/2] = $FS\left[\int_0^x |s|\,ds\right] = \frac{3}{2}\cdot 2\sum_{k=1}^{\infty} \frac{3(-1)^{k-1}\sin(k x\, pi/3)}{k\, pi} + \sum_{k=1}^{\infty} \frac{6x^3}{(k\pi)^3}\left[(-1)^k - 1\right] \sin\frac{k\pi x}{3}$

(b) $FS\left[\int_0^x (s^2-4)\,ds + \frac{8}{3}x\right] = + m + \sum_{k=1}^{\infty} \frac{32}{(k\pi)^3}(-1)^{k+1} \sin\frac{k\pi x}{2}$

$$m = \frac{1}{4}\int_{-2}^{2}\left(\frac{s^3}{3} - 4s\right)ds = \frac{1}{4}\left[\frac{s^4}{12} - 2s^2\right]_{-2}^{2} = 0$$

FS[x^3/3 - 4x] = $FS\left[\int_0^x (s^2-4)\,ds\right] = -\frac{8}{3}\cdot 2\sum_{k=1}^{\infty} \frac{2(-1)^{k-1}\sin(k x\, pi/2)}{k\, pi} + \sum_{k=1}^{\infty} \frac{32}{(k\pi)^3}(-1)^{k+1} \sin\frac{k\pi x}{2}$

FS[x] over [-2, 2]

(c) $FS\left[\int_0^x e^s\,ds - \frac{1}{20}\sinh 10x\right] = \sum_{k=1}^{\infty} \frac{200(-1)^k}{(10^2+k^2\pi^2)k\pi}\sinh 10 \sin\frac{k\pi x}{10}$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k\, 20k\pi}{k\pi(100+k^2\pi^2)}\sinh 10 \cos\frac{k\pi x}{10} + m$$

$$m = \frac{1}{20}\int_{-10}^{10} e^s\,ds = \frac{e^{10}-e^{-10}}{20} = \frac{1}{10}\sinh 10$$

(d) $FS\left[\int_0^x \sin s\,ds\right] = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}\sin 1}{1-k^2\pi^2}\cos k\pi x + m$

$$m = \frac{1}{2}\int_{-1}^{1} -\cos s\,ds = \left.-\frac{\sin s}{2}\right]_{-1}^{1} = \frac{-\sin 1 + \sin(-1)}{2}$$

$$= -\sin 1$$

FS[1-cos(x)] = $FS\left[\int_0^x \sin s\,ds\right] = -\sin 1 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}\sin 1}{1-k^2\pi^2}\cos k\pi x$

over [-10,10]

$$FS\left[\int_0^x e^s\,ds\right] = \frac{1}{20}\sinh 10\; FS[x] + \frac{1}{10}\sinh 10 + \sum_{k=1}^{\infty} \cdots + \sum_{k=1}^{\infty} \cdots$$

FS[e^x - 1] $= \frac{1}{10}\sinh 10 \sum_{k=1}^{\infty} \frac{10(-1)^{k-1}\sin(k x\, pi/10)}{k\, pi} + \frac{1}{10}\sinh 10 + \sum_{k=1}^{\infty} \cdots + \sum_{k=1}^{\infty} \cdots$

3.4.6 - $f(x)$ even on $[-l, l]$

$$FS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}$$

$$a_k = \frac{2}{l}\int_0^l f(x)\cos\frac{k\pi x}{l}\,dx, \quad k \geq 0$$

$$b_k = 0$$

$f(x)$ odd on $[-l, l]$

$$FS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}$$

$$a_k = 0$$

$$b_k = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi x}{l}\,dx$$

3.4.7 - a) $f(0) = f(l) = 0$

b) $f(0) = f(l) = 0$ & $f'(l) = f'(0) = 0$

3.4.8 - a) Consider $f(x)$ on $[0, 2\pi]$

Let $\hat{x} = x - \pi$

$f(x) = \hat{f}(\hat{x}) = f(\hat{x} + \pi)$ lies on $[-\pi, \pi]$

$$FS[\hat{f}(\hat{x})] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\hat{x} + b_k \sin k\hat{x}$$

$$FS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k(x-\pi)) + b_k \sin(k(x-\pi))$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (-1)^k a_k \cos kx + (-1)^k b_k \sin kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (-1)^k (a_k \cos kx + b_k \sin kx)$$

where:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}(\hat{x})\cos k\hat{x}\,d\hat{x} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(k(x-\pi))\,dx$$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$$\Rightarrow = \frac{1}{\pi}\int_0^{2\pi} (-1)^k f(x)\cos kx\,dx \qquad k = 0, 1, 2, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}(\hat{x}) \sin k\hat{x}\, d\hat{x}$$
$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x) \sin\big(k(x-\pi)\big)\, dx$$
$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\left[\sin kx \cos k\pi - \cos kx \sin k\pi\right] dx$$
$$= \frac{1}{\pi}\int_{0}^{2\pi} (-1)^k f(x) \sin kx\, dx \qquad k = 1, 2, \ldots$$

b) $f(x) = x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{(2\pi)^2}{2\pi} = 2\pi$$

$$a_k = \frac{(-1)^k}{\pi}\int_0^{2\pi} x\cos kx\,dx = \frac{(-1)^k}{\pi}\left[\frac{1}{k^2}\cos kx\right]_0^{2\pi}$$
$$= 0$$

x	$\cos kx$
1	$\frac{1}{k}\sin kx$
0	$-\frac{1}{k^2}\cos kx$

$$b_k = \frac{(-1)^k}{\pi}\int_0^{2\pi} x\sin kx\,dx = \frac{(-1)^k}{\pi}\left[\frac{-x}{k}\cos kx\right]_0^{2\pi}$$
$$= \frac{(-1)^k}{\pi}\cdot\frac{-2\pi}{k} = \frac{2(-1)^{k+1}}{k}$$

x	$\sin kx$
1	$-\frac{1}{k}\cos kx$
0	$-\frac{1}{k^2}\sin kx$

$$FS[x] = \pi + \sum_{k=1}^{\infty} (-1)^k \cdot \frac{2}{k}(-1)^{k+1}$$
$$= \pi - \sum_{k=1}^{\infty} \frac{2}{k}\sin kx$$

No, it is not the same result

3.4.9- $f(x) = x \;,\; 1 \le x \le 2 \qquad 2l = 1,\ l = \frac{1}{2}$

<u>Way 1</u>: Let $\hat{x} = x - \frac{3}{2}$

$f(x) = \hat{f}(\hat{x}) = f(\hat{x} + \frac{3}{2})$ lies on $[-\frac{1}{2}, \frac{1}{2}]$

$$FS[\hat{f}(\hat{x})] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2k\pi\hat{x} + b_k \sin 2k\pi\hat{x}$$
$$FS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2k\pi\left(x - \frac{3}{2}\right) + b_k \sin 2k\pi\left(x - \frac{3}{2}\right)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (-1)^k\left[a_k \cos 2k\pi x + b_k \sin 2k\pi x\right]$$

where:

$$a_k = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\hat{x}) \cos 2k\pi\hat{x}\, d\hat{x}$$

$$= 2\int_{1}^{2} f(x) \cos 2k\pi\left(x-\tfrac{3}{2}\right) dx$$

$$= 2\int_{1}^{2} (-1)^k f(x) \cos 2k\pi x\, dx \qquad k = 0, 1, 2, \ldots$$

$$b_k = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\hat{x}) \sin 2k\pi\hat{x}\, d\hat{x}$$

$$= 2\int_{1}^{2} f(x) \sin 2k\pi\left(x-\tfrac{3}{2}\right) dx$$

$$= 2\int_{1}^{2} (-1)^k f(x) \sin 2k\pi x\, dx$$

See that $\cos 2k\pi\left(x-\frac{3}{2}\right) = \cos\left(2k\pi x - 3k\pi\right)$

$$= \cos 2k\pi x \cos 3k\pi + \sin 2k\pi x \sin 3k\pi$$

$$= (-1)^k \cos 2k\pi x$$

Similarly, $\sin 2k\pi\left(x-\frac{3}{2}\right) = (-1)^k \sin 2k\pi x$

$$a_0 = 2\int_{1}^{2} x\, dx = x^2\Big]_1^2 = 3$$

$$a_k = 2\int_{1}^{2} (-1)^k x \cos 2k\pi x\, dx$$

$$= 2(-1)^k\,[0] = 0$$

$$\begin{array}{ll} x & \cos 2k\pi x \\ 1 & \frac{1}{2k\pi}\sin 2k\pi x \\ 0 & \frac{-1}{(2k\pi)^2}\cos 2k\pi x \end{array}$$

$$b_k = 2\int_{1}^{2} (-1)^k x \sin 2k\pi x\, dx$$

$$= 2(-1)^k\left[\frac{-x}{2k\pi}\cos 2k\pi x\right]_1^2$$

$$= 2(-1)^k\left[\frac{-2+1}{2k\pi}\right] = \frac{(-1)^{k+1}}{k\pi}$$

$$\begin{array}{ll} x & \sin 2k\pi x \\ 1 & \frac{-1}{2k\pi}\cos 2k\pi x \\ 0 & \frac{-1}{(2k\pi)^2}\sin 2k\pi x \end{array}$$

$$FS[x] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\pi}(-1)^k \sin 2k\pi x + \frac{3}{2} = \frac{3}{2} - \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin 2k\pi x$$

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Way 2: $2l = b - a = 2 - 1 = 1$

$l = \frac{1}{2}$

$$\tilde{f}(x + 2lk) = f(x) \quad , \; 1 < x < 2$$

$$\tilde{f}(x) = \begin{cases} f(x-k) = x - k & 1 < x-k < 2 \\ & 1+k < x < 2+k \\ \frac{1+2}{2} = 1.5 & x = k \quad\quad k \in \mathbb{Z} \end{cases}$$

$$FS[\tilde{f}(x)] = FS[f(x)] = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi k x + b_k \sin 2\pi k x$$

$$a_0 = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(x)\, dx = 2\int_{-\frac{1}{2}}^{0} (x+2)\, dx + 2\int_{0}^{\frac{1}{2}} (x+1)\, dx$$

$$= 2\left[\frac{x^2}{2} + 2x\right]_{-\frac{1}{2}}^{0} + 2\left[\frac{x^2}{2} + x\right]_{0}^{\frac{1}{2}}$$

$$= \frac{7}{4} + \frac{5}{4} = 3.$$

$$a_k = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(x) \cos(2\pi k x)\, dx = 2\int_{-\frac{1}{2}}^{0} (x+2)\cos(2\pi k x)\, dx$$

$$+ 2\int_{0}^{\frac{1}{2}} (x+1) \cos(2\pi k x)\, dx$$

$$= 2\left[\int_{-\frac{1}{2}}^{0} x \cos 2\pi k x\, dx + \int_{-\frac{1}{2}}^{0} 2 \cos 2\pi k x\, dx + \int_{0}^{\frac{1}{2}} x \cos 2\pi k x + \int_{0}^{\frac{1}{2}} \cos 2\pi k x\, dx\right]$$

$$= 2\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x \cos 2\pi k x}_{\text{odd}}\, dx\right] + 4\int_{-\frac{1}{2}}^{0} \cos 2\pi k x\, dx + 2\int_{0}^{\frac{1}{2}} \cos 2\pi k x\, dx$$

$$= 0 + 4\left[\frac{1}{2\pi k} \sin 2\pi k x\right]_{-\frac{1}{2}}^{0} + 2\left[\frac{1}{2\pi k} \sin 2\pi k x\right]_{0}^{\frac{1}{2}}$$

$$= 0 + 0 + 0 = 0$$

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$$b_k = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(x)\,\sin(2\pi kx)\,dx$$

$$= 2\left[\int_{-\frac{1}{2}}^{0} (x+2)\sin 2\pi kx\,dx + \int_{0}^{\frac{1}{2}} (x+1)\sin 2\pi kx\,dx\right]$$

look at a_k (similarly)

$$= 2\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x\sin 2\pi kx}_{\text{even}}\,dx\right] + 4\int_{-\frac{1}{2}}^{0}\sin 2\pi kx\,dx + 2\int_{0}^{\frac{1}{2}}\sin 2\pi kx\,dx$$

$$= 4\int_{0}^{\frac{1}{2}} x\sin 2\pi kx\,dx - \frac{2}{\pi k}\Big[\cos 2\pi kx\Big]_{-\frac{1}{2}}^{0} - \frac{1}{\pi k}\Big[\cos 2\pi kx\Big]_{0}^{\frac{1}{2}}$$

$$= 4\int_{0}^{\frac{1}{2}} x\sin 2\pi kx\,dx - \frac{2}{\pi k}(1-(-1)^k) - \frac{1}{\pi k}((-1)^k - 1)$$

$$= 4\left[\frac{-x}{2\pi k}\cos 2\pi kx\right]_{0}^{\frac{1}{2}} \quad - - - -$$

$$\begin{array}{ll} x & \sin 2\pi kx \\ 1 & \frac{-1}{2\pi k}\cos 2\pi kx \\ 0 & \frac{-1}{(2\pi k)^2}\sin 2\pi kx \end{array}$$

$$= \frac{\cancel{4}(-1)^{k+1}}{\cancel{4}\pi k} + (1-(-1)^k)\left[\frac{1}{\pi k} - \frac{2}{\pi k}\right]$$

$$= \frac{(-1)^{k+1}}{\pi k} - \frac{1}{\pi k}(1-(-1)^k)$$

$$= \frac{(-1)^{k+1} - 1 + (-1)^k - 1}{\pi k} = \frac{-1}{\pi k}$$

$$FS[x] = \frac{3}{2} + \sum_{k=1}^{\infty} \frac{-1}{\pi k}\sin 2\pi kx$$

3.5.4 - $v_n(x) = \begin{cases} 1 & 0<x<\frac{1}{n} \\ 0 & \end{cases}$

$\lim_{n\to+\infty} v_n(x) = 0 \Rightarrow v_n$ conv pointwise to 0.

$\max |v_n(x)| = 1 \quad \forall n \in \mathbb{N}$.

$\Rightarrow v_n$ does not conv uniformly to 0.

3.5.6. $v_n(x) = nxe^{-nx^2}$

$\lim_{n\to+\infty} v_n(x) = \lim_{n\to+\infty} \frac{nx}{e^{nx^2}} = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow v_n$ conv pointwise to 0.

$v_n'(x) = ne^{-nx^2} - 2n^2x^2e^{-nx^2}$

$= (n-2n^2x^2)e^{-nx^2}$

$= 0 \Leftrightarrow n - 2n^2x = 0$

$x^2 = \frac{n}{2n^2} = \frac{1}{2n}$

$v_n\left(\left(\frac{1}{2n}\right)^{1/2}\right) = n\left(\frac{1}{2n}\right)^{1/2} e^{-n\left(\frac{1}{2n}\right)} = n\left(\frac{1}{2n}\right)^{1/2} e^{-\frac{1}{2}}$

$\max|v_n(x)| = n^{1/2}\frac{1}{2^{1/2}}e^{-\frac{1}{2}}$

$\lim_{n\to+\infty} \max|v_n(x)| \neq 0$.

v_n does not conv uniformly to 0.

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3.5.3. a) $\lim_{n\to+\infty} 1-\frac{x^2}{n^2} = 1$ Yes

b) $\lim_{n\to+\infty} e^{-nx} = \begin{cases} +\infty & x<0 \\ 1 & x=0 \\ 0 & x>0 \end{cases}$ No

c) $\lim_{n\to+\infty} e^{-nx^2} = \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$ Yes

d) $\lim_{n\to+\infty} |x-n| = +\infty$ No

e) $\lim_{n\to+\infty} \frac{1}{1+(x-n)^2} = 0$ Yes

f) $\lim_{n\to+\infty} f_n(x) =$ [illegible] Yes

g) $\lim_{n\to+\infty} f_n(x) = 0$ Yes

h) $\lim_{n\to+\infty} f_n(x) = x$ Yes

3.5.5.

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(a) $f_n(x) = \dfrac{-x^2}{n^2}$

$\lim_{n\to+\infty} f_n(x) = 0$

$$\max |f_n(x)| = \text{infinity}$$

$$\lim_{n\to+\infty} \max |f_n(x)| = \text{infinity} \Rightarrow f_n \to 0 \text{ non uniformly}$$

(b) $$f_n(x) = e^{-n|x|} = \begin{cases} e^{-nx} & x \geq 0 \\ e^{nx} & x \leq 0 \end{cases}$$

$$\lim_{n\to+\infty} f_n(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} = f(x)$$

$$|f_n(x) - f(x)| = \begin{cases} 0 & , \quad x = 0 \\ e^{-n|x|} & , \quad x \neq 0 \end{cases}$$

Since $0 < e^{-n|x|} < 1$ for $\cancel{x\neq} 0$, then

$$\max |f_n(x) - f(x)| = \max e^{-n|x|} \approx 1$$

Thus $f_n(x)$ does not converge uniformly to f.

ⓒ $f_n(x) = xe^{-n|x|} = \begin{cases} xe^{-nx} & x \geq 0 \\ xe^{nx} & x \leq 0 \end{cases}$

$\lim\limits_{n \to +\infty} f_n(x) = 0$

$f_n'(x) = \begin{cases} e^{-nx} - nxe^{-nx} & x > 0 \\ e^{nx} + nxe^{nx} & x < 0 \end{cases}$

$f_n'(x) = 0 \iff 1 - nx = 0 \qquad 1 + nx = 0$

$x = \frac{1}{n} \qquad x = -\frac{1}{n}$

$|f_n(x)| \leq \frac{1}{n} e^{-1} \xrightarrow[n \to +\infty]{} 0$

$f_n \to 0$ uniformly.

(d) $f_n(x) = \dfrac{1}{n(1+x^2)}$

$\lim_{n \to +\infty} f_n(x) = 0$

$f_n'(x) = \dfrac{-2nx}{n^2(1+x^2)^2} = 0 \iff x = 0.$

$f_n(0) = \dfrac{1}{n}$

$|f_n(x)| \leq \dfrac{1}{n} \xrightarrow[n \to +\infty]{} 0$

$\therefore$ $f_n(x)$ conv uni to 0

(e) $f_n(x) = \dfrac{1}{1+(x-n)^2}$

$\lim_{n \to +\infty} f_n(x) = 0$

$f_n'(x) = \dfrac{-2(x-n)}{[1+(x-n)^2]^2} = 0 \iff x = n$

$f_n(n) = 1$

$|f_n(x)| \leq 1$

$\lim_{n \to +\infty} \max |f_n(x)| = 1 \neq 0$

f_n doesnot conv uni to 0.

(f) $f_n(x) = |x-n|$ doesnot conv at all

$\lim_{n \to +\infty} f_n(x) = +\infty$

(g) $f_n(x) = \begin{cases} \frac{1}{n} & 0 < |x| < n \\ 0 & — \end{cases}$

$\lim_{n \to +\infty} f_n(x) = 0$

$\max |f_n(x)| = \frac{1}{n}$

$\lim_{n \to +\infty} \max |f_n(x)| = 0$ (obvious)

$f_n \to 0$ uniformly

(h) $f_n(x) = \begin{cases} n & 0 < |x| < \frac{1}{n} \\ 0 & — \end{cases}$

$\lim_{n \to +\infty} f_n(x) = 0$

$\max |f_n(x)| = n \xrightarrow[n \to +\infty]{} +\infty$

f_n does not conv uniformly to 0

(i) $f_n(x) = \begin{cases} \frac{x}{n} & |x| < 1 \\ \frac{1}{nx} & |x| \geq 1 \end{cases}$

$\lim_{n \to +\infty} f_n(x) = 0$

$\max |f_n(x)| = \frac{1}{n} \xrightarrow[n \to +\infty]{} 0$

f_n conv unif to 0.